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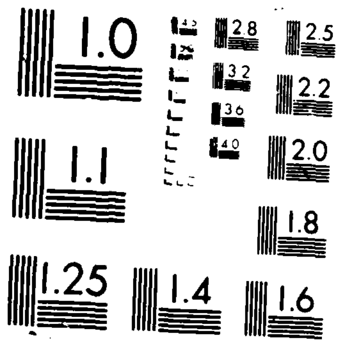


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# CENTER FOR STOCHASTIC PROCESSES

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POINT PROCESSES IN THE PLANE

by

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# POINT PROCESSES IN THE PLANE

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Abstract: In this survey paper, two-parameter point processes are studied in connection with martingale theory and with respect to the partial-order induced by the Cartesian coordinates of the plane. Point processes are characterized by jump stopping times and by their two-parameter compensators. Properties of the doubly stochastic Poisson process, as predictability, are discussed. A definition for the Palm measure of a two-parameter stationary point process is proposed.

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Keywords and Phrases: Point process, martingale, partial-order, predictability, stopping line, optional increasing path, compensator, intensity, stationarity, orderliness, doubly stochastic Poisson process, Palm measure.



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## 0. Introduction

A point process in the plane is a random distribution of points in a subspace of the plane, generally the positive quadrant $\mathbb{R}_+^2$. Whereas the point processes on the real line have particular properties derived from the natural linear order of the real numbers, the case of plane or generally $\mathbb{R}^n$-point processes seems more difficult due to the lack of total order between the jump points of the process. Our main interest here is to study the properties of the point processes derived by the partial-order structure in the plane. Here we treat only the two-parameter case (the plane), but almost every result can be simply extended to the n-parameter case, following the natural partial-order in $\mathbb{R}^n$, and sometimes to lattices or general directed sets as it was begun by Mazziotto and Merzbach in [19].

The general case in which the parameter set is a $\sigma$-algebra of subsets of some space was first studied extensively by J.F.C. Kingman [15] and by J. Mecke. Some developments are due to Y.K. Belyayev, M.R. Leadbetter [16], R.K. Milne [30], P. Jagers [12], O. Kallenberg [14], and J. Neveu [31].

This paper is essentially a survey paper, especially on the author's papers on the subject. But clearly, several interesting directions are not treated at all here, for example the Poisson calculus (see Mazziotto-Szpirglas [21]) and Markov properties which are understudied up to today (see [19]).

We hope that the techniques developed here could be applied to control theory, multi-armed bandit problems, random geometry (ecology, astronomy), multi-components machine problems and queueing theory. For example a two-server queueing process $Q_z$ can be well described by $Q_z = Q_0 + A_z - D_z$ where $Q_0$ is the initial state and $A_z$ and $D_z$ are point processes. For each $z = (s,t)$, the random variable $Q_z$ can be interpreted as the number of customers waiting in the first line at time s and waiting in the second line at time t. This kind of problem occurs where the two lines (or servers) are not in proximity one to the other, and we don't obtain information from the lines at the same time. The process $A_z(D_z)$ is the number of arrivals (departures) in the rectangle $R_z$, and is called the arrival (departure) process.

The paper is divided as follows. In the first section we develop the basic tools for the dynamical study of two-parameter processes such as the notions of predictability, stopping lines and optional increasing paths. Point processes are defined in the second section. We study simple and strictly simple point processes, compensation (dual predictable projection) and characterizations of the jump lines associated with a point process. We present also conditions in order to obtain strong martingales, and, as example, the one-jump process. The third section is devoted to the concepts of measures, stationarity and orderliness with its ramifications, extending some works of Daley [7]. We treat general results such as Korolyuk's theorem and Dobrushin's lemma following the approach of M.R. Leadbetter [16]. The notion of stationarity is introduced. Section four is entirely devoted to the (doubly stochastic) Poisson process. We put here several characterizations and this process is a good illustration of the previous sections.

In the last section, we define and study the Palm measure (see [24]). For one-parameter simple point processes, it was introduced for the first time by Palm (in 1943) and studied by A.Y. Khinchin. In the sixties and seventies the Palm measure was considerably extended by Ryll-Nardzewski, K. Matthes, J. Mecke [23], P. Jagers (see [12]), and others, in particular for point processes on a locally compact space with a countable basis. A good account of today's theory is given by J. Neveu [31]. Here we will present a new definition for the two-parameter case, which extends the classical definition on the line which obtains the Palm measure by the translation of the probability of the greatest jump point which is smaller than the origin. For this purpose, the main idea is to look at the jump lines, instead of the jump points, associated with a point process, in order to obtain a well-ordered increasing sequence of jumps. This new and fruitful technique was already used by Mazziotto-Merzbach [19] in other connections.

I wish to take the opportunity to thank the faculty members and the staff of the Center for Stochastic Processes at Chapel Hill for its warm hospitality, which permitted me to write this paper in a quiet and mathematically encouraging atmosphere.

## 1. Notation and general background

The usual notation and the main tools are introduced as follows: The processes are indexed by points of $\mathbb{R}_+^2$ in which the partial order induced by the Cartesian coordinates is defined: let $z = (s,t)$ and $z' = (s',t')$, then $z \le z'$ if $s \le s'$ and $t \le t'$, and $z < z'$ if $s < s'$ and $t < t'$. We denote $z \wedge z'$ if $s \le s'$ and $t \ge t'$. A probability space $(\Omega, F, \mathbb{P})$ is given equipped with an increasing right-continuous filtration $\{F_z, z \in \mathbb{R}_+^2\}$ of sub-$\sigma$-algebras of $F$. For $z = (s,t)$, denote $F_z^1 = F_{(s,\infty)}$ and $F_z^2 = F_{(\infty,t)}$ and $F_z^* = F_z^1 \vee F_z^2$. The conditional independence property, for every $z$, $F_z^1$ and $F_z^2$ are conditionally independent given $F_z$, will not be assumed throughout.

Denote by $S$ the set of all the decreasing lines, i.e. $L \in S$ if and only if

(i) For all $z,z' \in L \Rightarrow$ either $z \wedge z'$ or $z' \wedge z$.

(ii) For all $z \in \mathbb{R}_+^2$ and $z \notin L$, $\exists z' \in L : z < z'$ or $z' < z$.

For each $z = (s,t)$, denote $\bar{z} = \{(s,t') : t \le t'\} \cup \{(s',t) : s \le s'\}$, $\underline{z} = \{(s,t') \}: t' \le t\} \cup \{(s',t) : s' \le s\}$ and $\bar{\underline{z}} = \bar{z} \cup \underline{z}$. Clearly $\bar{z}, \underline{z} \in S$ (but not $\bar{\underline{z}}$). If $L,L' \in S$, we denote $L \le L'$ if for all $z \in L$, $\exists z' \in L'$ such that $z \le z'$. This relation defines a partial order in $S$. $L < L'$ will mean $L \le L'$ and $L \cap L' = \phi$. Also $z \le L$ will mean $\underline{z} \le L$.

$L \wedge L' = \sup\{L'' : L'' \le L$ and $L'' \le L'\}$

$L \vee L' = \inf\{L'' : L \le L''$ and $L' \le L''\}$.

Let $A$ be a subset of $\mathbb{R}_+^2$, the Debut of $A$, denoted $D_A$ will be the greatest element of $S$ such that $z < D_A \Rightarrow z \notin A$. (For example, $D_{\{z\}} = \bar{z}$.)

A random decreasing line $L : \Omega \to S$ is called a stopping line if for every $z \in \mathbb{R}_+^2$, $\{\omega : z \le L(\omega)\} \in F_z$. A stopping point $Z$ is a random point such that $\bar{Z}$ is a stopping line. L is called a stepped stopping line if for every $\omega \in \Omega$, the set of the minimal points of $L(\omega)$ is denumerable and is finite in every bounded domain. A random increasing path $\Gamma$ is called an optional increasing path if for every stopping line L, $D_{L\cap\Gamma}$ is a stopping point.

In the product space $\Omega \times \mathbb{R}_+^2$, the predictable (resp. 1-predictable, 2-predictable, *-predictable) $\sigma$-algebra is defined to be the $\sigma$-algebra generated by the sets $F \times (z,z']$, where $F \in \mathcal{F}_z$ (resp. $F \in \mathcal{F}_z^1$, $F \in \mathcal{F}_z^2$, $F \in \mathcal{F}_z^*$) and $(z,z']$ is the rectangle $\{\xi : z < \xi \le z'\}$; it is denoted $\mathcal{P}$ (resp. $\mathcal{P}^1, \mathcal{P}^2, \mathcal{P}^*$). In $\Omega \times \mathbb{R}_+^2 \times \mathbb{R}_+^2$, another predictable $\sigma$-algebra is needed: $\hat{\mathcal{P}}$ is defined to be the $\sigma$-algebra generated by the sets $F \times (z_1,z_1'] \times (z_2,z_2']$ where $F \in \mathcal{F}_{\sup(z_1,z_2)}$, and every couple taken from $(z_1,z_1'] \times (z_2,z_2']$ satisfies the relation $\wedge$. A stopping line L is called predictable or announcable if its graph $[\![L]\!] = \{(\omega,z) : z \in L(\omega)\}$ is a predictable set.

A process $A = \{A_z, z \in R_+^2\}$ is called increasing if its increment on every rectangle $(z,z']$ is nonnegative:
$A(z,z'] = A_{z'} - A_{(s,t')} - A_{(s',t)} + A_z \ge 0$. The difference of two increasing processes is called a process of bounded variation. Let us introduce the different kinds of martingales used below. Let $M = \{M_z, z \in R_+^2\}$ be an adapted and integrable process. M is a weak martingale if $E[M(z,z']/\mathcal{F}_z] = 0$, M is an i-martingale if $\mathcal{F}_z$ is replaced by $\mathcal{F}_z^i$ $(i=1,2)$, M is a martingale if $E[M_{z'}/\mathcal{F}_z] = M_z$ (which is equivalent to say, where the conditional independent property is satisfied, that it is a 1-martingale and a 2-martingale and a one-parameter martingale on the axes), and M is a strong martingale if it

is a martingale and $E[M(z,z']/F^*_z] = 0$, for every $z < z'$ in $\mathbb{R}^2_+$ (see [25]). To every increasing integrable and adapted process A, we can associate its dual predictable (resp. i-predictable, $i = 1,2$) projection denoted $A^\pi$ (resp. $A^{(i)}$, $i = 1,2$ ). It is characterized to be the unique predictable (resp. i-predictable, $i = 1,2$) increasing process such that $A - A^\pi$ (resp. $A - A^{(i)}$, $i = 1,2$) is a weak martingale (resp. i-martingale, $i = 1,2$) [28]. Let $X = \{X_z, z \in \mathbb{R}^2_+\}$ be a right-continuous process ($\lim\limits_{\substack{z<z' \\ z'\to z}} X_{z'} = X_z$) possessing limits in the other quadrants, and denote its jump at $z = (s,t)$ by the following: $\Delta X_z = X_z - X_{(s^-,t)} - X_{(s,t^-)} + X_{z^-}$, $\Delta^1 X_z = X_z - X_{(s^-,t)}$ and $\Delta^2 X_z = X_z - X_{(s,t^-)}$. Therefore $\Delta X_z = \Delta^1 X_z - \Delta^1 X_{(s,t^-)} = \Delta^2 X_z - \Delta^2 X_{(s^-,t)}$. Moreover, if X is increasing then the set of its discontinuous points is constituted by a countable number of semi-lines parallel to the axes and if X is also adapted then this set is a countable union of stepped stopping lines [20].

## 2. Point processes

Definition: A right-continuous process $M = \{M_z, z \in \mathbb{R}_+^2\}$ is called a plane point process if

(i) $M$ vanishes on the axes and takes its values in $\mathbb{N} \cup \{\infty\}$,

(ii) $M$ is increasing,

(iii) $M$ is adapted (with respect to a given filtration $\{F_z, z \in \mathbb{R}_+^2\}$).

In [27] we required also that for every $z \in \mathbb{R}_+^2$, $\Delta M_z, \Delta^1 M_z, \Delta^2 M_z \in \{0,1\}$. Here, a process satisfying this property will be called strictly simple, and if we require only $\Delta M_z \in \{0,1\}$, the process will be called simple. It is clear that if for every $z \in \mathbb{R}_+^2$, $\Delta^1 M_z, \Delta^2 M_z \in \{0,1\}$, then $M$ is strictly simple. For all $z$, we have $M_z = \Sigma_{z' \leq z} \Delta M_{z'}$, therefore $M$ can be characterized as an adapted discrete measure which is a linear combination of Dirac measures $\Sigma_n \alpha_n \varepsilon_{Z_n}$ on the random jump points $\{Z_n\}$, e.g. the set of the (different) points such that $\Delta M_{Z_n} \neq 0$.

To every point process $M$, we can associate another point process $M^*$ which is simple, defining it by $\Sigma_n \varepsilon_{Z_n}$. Notice that $M^*$ is not necessarily strictly simple, but, as a consequence of the following proposition, we can associate to $M$ another point process $M^{**}$, which is strictly simple, by deleting for every $n$ the jump point $Z_n$ which belongs to vertical or horizontal lines generated by $\{Z_m, m < n\}$.

Proposition 2.1: $M$ is a strictly simple point process if and only if $\mathbb{P}\{M(L) = 0 \text{ or } 1 \text{ for every segment } L \text{ parallel to one of the axes}\} = 1$.

Proof: Suppose $M$ is strictly simple, and let $L$ be a segment parallel to say, the first axis and that $M(L) > 1$. Then there are at least two consecutive points $z = (s,t)$ and $z' = (s',t)$ on $L$ which are jump points $\Delta M_z = \Delta M_{z'} = 1$. Since $M$ is strictly simple, we obtain $\Delta^1 M_{z'} = \Delta^2 M_{z'} = 1$ and therefore $M_{(s'^-,t)} = M_{(s',t^-)} = M_{(s'^-,t^-)}$. We can suppose that $s < s'$. If $M$ is constant on the interval $[s,s')$, then there are no jump points in $((s,0),(s',0))$. Thus $M_{(s'^-,t)} = M_{(s,t)}$ and $M_{(s'^-,t^-)} = M_{(s,t^-)}$ which means that $M_z = M_{(s,t^-)}$. This contradicts the fact that $\Delta^2 M_z = 1$.

Conversely, suppose the condition of the proposition is verified and suppose that there exists a point $z$ such that $\Delta^1 M_z > 1$. Then, at least one of the segments of $\underline{z}$ has a M-measure greater than 1, which contradicts the given condition.

Coming back to the jump points $\{Z_n\}_n$ of the point process $M$, note that they are not, in general, stopping points and therefore we cannot expect to characterize $M$ by its jump points. However, if $M$ is strictly simple, the jump points $\{Z_n\}_n$ are characterized by the following properties:

(i) $Z_0 = (0,0)$ and if $Z_n = \infty$ then $Z_m = \infty$ for all $m > n$.

(ii) For all $n$ such that $Z_n < \infty$ then $Z_m \not\leq Z_n$ for all $m > n$.

(iii) For all $n \geq 1$, $\Delta M_{Z_n} = 1$ a.s. and $M_z = \sum_n I_{\{Z_n \leq z\}}$.

(iv) For every random point $Z$ such that $[Z] \cap (\cup_n [Z_n])$ is evanescent, we have $\Delta M_Z = 0$ a.s., and if moreover $[Z] \cap (\cup [\overline{Z}_n])$ is evanescent, $\Delta^1 M_Z = \Delta^2 M_Z = 0$ a.s. (This condition means that if $\Delta^i M_Z = 1$ then there exists an integer $n$ such that $Z \in \overline{Z}_n$ and if moreover $\Delta^{3-i} M_Z = 1$, then there exist integers $m$ and $n$ such that $Z = \overline{Z}_n \cap \overline{Z}_m$.)

Conversely, let $\{Z_n\}_n$ be a sequence of stopping points satisfying (i) and (ii). Then the process $M$ defined by $M_z = k - 1$ where $k$ is the number of sets $[Z_n, \infty)$ which contain the point $z$, is the strictly simple point process $\{Z_n\}_n$.

In the same spirit, we can define the concept of multivariate plane point process using the notion of discrete measure, since the couples of random variables $(Z_n, X_n)_{n\geq 1}$ cannot characterize a multivariate point process. We consider a Lusin space E and an extra point $\Delta$. A multivariate point process is the following discrete random measure on $\mathbb{R}_+^2 \times E$:

$$\mu(\omega; dz, dx) = \sum_{n\geq 1} I_{\{Z_n(\omega) < \infty\}} \cdot \varepsilon_{(Z_n(\omega), X_n(\omega))}(dz, dx),$$

- where $\varepsilon_a$ denotes the Dirac measure located at point a,
- the random points $\{Z_n\}_n$ satisfy properties (i) and (ii) from above,
- $\{X_n\}_n$ are random variables in $E \cup \{\Delta\}$,
- $X_n(\omega) = \Delta$ if and only if $Z_n(\omega) = \infty$,

For each Borel subset C of E, the process

$$M_z(C) = M(\mathbb{R}_z \times C) = \sum_{n\geq 1} I_{\{Z_n \leq z\}} \cdot I_C(X_n) \text{ is adapted.}$$

Note that if E reduces to one point, then $M_z(E)$ reduces to an ordinary strictly simple point process.

As in the one-parameter case, we can prove and characterize the existence of the dual predictable projection of a multivariate point process [27].

Let us now introduce the following sequences of random lines associated with a given point process M. Define $L_1 = L_1' = D_{\{M_z \geq 1\}} = \Lambda_n \overline{Z}_n$, and for $n > 1$, define $L_n = D_{\{z : \Delta M_z = 1, L_{n-1} \prec z\}}$ (which is equal to $\Lambda_k \overline{Z}_k$ for all the integers k such that $L_{n-1} < \overline{Z}_k$) and $L_n' = D_{\{M_z \geq n\}}$.

<u>Proposition 2.2</u>: Any of the sequences $\{L_n\}_{n=1}^{\infty}$ or $\{L'_n\}_{n=1}^{\infty}$ which satisfy the respective following properties characterize the strictly simple point process M.

(i) For all n, $L_n$ and $L'_n$ are stepped stopping lines,

(ii) The sequences $\{L_n\}_n$ and $\{L'_n\}_n$ are increasing,

(iii) $\{L_n\}_n$ is disjoint: $[L_n] \cap [L_m] = \phi$ for $m \neq n$,

(iv) For all $m \neq n$, for all $\omega \in \Omega$, the set $[L'_n(\omega)] \cap [L'_m(\omega)]$ is countable. Moreover these lines satisfy

$$\cup_n [Z_n] \subset \cup_n [L_n] = \cup_n [\bar{Z}_n] \subset \cup_n [L'_n].$$

<u>Proof</u>: [27] Let the sequence $\{L'_n\}_n$ satisfying (i), (ii), (iv) be given and construct the following bounded variation process:

$$B_z = \begin{cases} 0 \text{ if } z < L'_1 \\ n \text{ if } L'_n \le z < L'_{n+1} \\ \infty \text{ if for all } n,\ L'_n < z. \end{cases}$$

This process is adapted and can be decomposed by $B = M - N$ where $M_z = \Sigma_{z' \le z} I_{\{\Delta B_{z'} = +1\}}$, $N_z = -\Sigma_{z' \le z} I_{\{\Delta B_{z'} = -1\}}$ are adapted and increasing processes. M is the process associated to the sequence $\{L'_n\}_n$. The same holds for the sequence $\{L_n\}_n$.

<u>Proposition 2.3</u>: If M is a strictly simple point process such that every random point Z satisfying $\Delta^1 M_Z + \Delta^2 M_Z - \Delta M_Z = 2$ is not a stopping point. Then for any optional increasing path $\Gamma$, the one-parameter point process along this path $M^{\Gamma}$ is also simple. Conversely, if for any optional increasing path $\Gamma$, the one-parameter process $M^{\Gamma}$ is a simple point process and suppose that M is increasing, then M is a strictly simple point process.

The proof of this proposition is easy since the first point of intersection between an optional increasing path and a stopping line is a stopping point. This proposition shows that the strictly simple property is very natural when we extend the simple property from the one-parameter case.

Turning now to the properties of predictability for a point process $M$, let us call the dual predictable projection of $M$ the compensator of $M$, denoted $\bar{M}$ or the predictable measure associated with $M$, and if this measure is absolutely continuous with respect to the Lebesgue measure, denote the density by $\lambda_z$ and call it the intensity of the point process $M$. This process can be chosen to be predictable. Notice that $M - M^{(1)} - M^{(2)} + M^{\pi}$ is a martingale of bounded variation. We have to note here that it was proved in [19] that even in the general case where the parameter set is a directed set, every point process can be described as an integer measure on a well-ordered set, and then the predictable projection and the dual predictable projection of a point process can be constructed.

Theorem 2.4: Let $M$ be a simple process whose compensator $\bar{M}$ is continuous and $M - \bar{M}$ is a strong martingale. Then $M$ is strictly simple and more generally, with probability one, any given optional increasing path contains at most one jump point.

Proof: The main idea of the proof follows Ivanoff [10], except the fact that $\bar{M}$ must not necessarily be deterministic. For $K < \infty$ arbitrary, define a rectangular grid $\{D_{ij}^{(n)}\}$ of $[0,K]^2$, which mesh tends to zero where $n$ tends to infinity. Let $\Gamma$ be an optional increasing path, $A$ the event that $\Gamma$ contains more than one point, and $B_n$ the event that $M(D_{ij}^{(n)}) > 1$ for some pair $(i,j)$. Therefore

$$A \subseteq \bigcup_{i,j} \bigcup_{\substack{(k,\ell)\geq(i,j)\\(k,\ell)\neq(i,j)}} \{M(D_{i,j}^{(n)}\cap\Gamma) > 0\} \cap \{M(D_{k,\ell}^{(n)}\cap\Gamma) > 0\} \cup B_n$$

and

$$\mathbb{P}(A) \leq \sum_{i,j} \sum_{\substack{(k,\ell)\geq(i,j)\\(h,\ell)\neq(i,j)}} \mathbb{P}\{M(D_{h,\ell}^{(n)}\cap\Gamma) > 0 \mid M(D_{i,j}^{(n)}\cap\Gamma) > 0\}\,\mathbb{P}\{M(D_{i,j}^{(n)}\cap\Gamma) > 0\} + \mathbb{P}(B_n).$$

Since $M$ is simple, $\mathbb{P}(B_n) \to 0$ as $n \to \infty$. Now, note that $\{M(D_{i,j}^{(n)}\cap\Gamma) > 0\} \in F^*_{(k,\ell)}$ for any pair $(k,\ell) \geq (i,j)$, $(k,\ell) \neq (i,j)$. Thus for any $\varepsilon > 0$, if $n$ is sufficiently large,

$$\mathbb{P}\{M(D_{k,\ell}^{(n)}\cap\Gamma) > 0 \mid M(D_{i,j}^{(n)}\cap\Gamma) > 0\} \leq E[M(D_{k,\ell}^{(n)}\cap\Gamma) \mid M(D_{i,j}^{(n)}\cap\Gamma) > 0]$$

$$= E[\overline{M}(D_{k,\ell}^{(n)}\cap\Gamma) \mid M(D_{i,j}^{(n)}\cap\Gamma) > 0] \leq \varepsilon.$$

Finally, for $n$ sufficiently large, we obtain

$$\mathbb{P}(A) \leq \varepsilon \sum_{i,j} \sum_{k,\ell} \mathbb{P}\{M(D_{i,j}^{(n)}\cap\Gamma) > 0\} + \mathbb{P}(B_n),$$

and therefore

$$\mathbb{P}(A) = 0.$$ □

The following proposition was proved by Ivanoff [10].

Proposition 2.5: Let $M$ be a simple point process and suppose that the conditional independence property is satisfied. Then $M - \overline{M}$ is a strong martingale.

Let us mention also the following result due to Kallenberg [13], despite the fact that it is a general result in which the partial order does not intervene.

Proposition 2.6: Let $M$ be a point process and $\mathcal{A} \subseteq \mathcal{B}$ be an algebra containing some basis for $\mathbb{R}_+^2$. Then the distribution of $M^*$ is uniquely determined by all $\mathbb{P}\{M(A) = 0\}$ for bounded $A \in \mathcal{A}$.

Recall that for any non-negative process $X = \{X_z, z \in \mathbb{R}_+^2\}$, the predictable projection of X is defined to be the unique predictable process $Y = \{Y_z, z \in \mathbb{R}_+^2\}$ such that $E[\int X_z dA_z] = E[\int Y_z dA_z]$ for every increasing and predictable process $A = \{A_z, z \in \mathbb{R}_+^2\}$. The following proposition was essentially proved in [27].

Proposition 2.7: Let M be a point process, $Z_n$ its jump points and $L_n$ and $L_n'$ its associated jump lines. Assume the following statements:

(i) M is a predictable process,

(ii) The stopping lines $\{L_n\}_n$ are predictable (announcable),

(iii) The stopping lines $\{L_n'\}_n$ are predictable (announcable),

(iv) For all n, the predictable projection of the process $I_{[\![L_n']\!]}$ vanishes,

(v) M is a quasi-continuous to the left: for every predictable stopping line L, $\int I_{[\![L]\!]} dM = 0$.

(vi) For every predictable stepped stopping line L, $\int I_{[\![L]\!]} dM = 0$,

(vii) The dual predictable projection $M^\pi$ is a continuous process,

(viii) The predictable projection of the process $I_{[\![Z_n]\!]}$ vanishes.

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) and (iv) $\Rightarrow$ (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii) $\Rightarrow$ (viii).

Returning to the compensator $\bar{M}$, it is the unique predictable increasing process such that $M - \bar{M}$ is a weak martingale, and can be calculated directly as follows.

Proposition 2.8: Let M be a simple point process. Then for every $z \in \mathbb{R}_+^2$,

$$\bar{M}_z = \lim_{n\to\infty} \sum_i E[M(D_i^{(n)}) \mid F_{di,n}] = \lim_{n\to\infty} \sum_i \mathbb{P}\{M(D_i^{(n)}) > 0 \mid F_{di,n}\}$$

where for every n, $\{D_i^{(n)}\}_i$ is a rectangle partition of the rectangle $[(0,0),z]$, $d_{i,n}$ is the first point of $D_i^{(n)}$ and it is assumed that the mesh size of the $n^{th}$ partition tends to zero.

Example: There are two kinds of diametrically opposite examples, the Poisson process that we shall study in detail later and the "one-jump process" summed up here.

One-jump process: This kind of process was extensively studied by A. Al-Hussaini and R. Elliott [1,2,3] and also by G. Mazziotto and J. Szpirglas [22]. Let $Z = (S,T)$ be a stopping point and consider the point process $M = I_{[Z,\infty)}$. Denote by F the distribution function of Z and by G its survivor function: $G(z) = \mathbb{P}\{z < Z\}$. The predictable measure of M clearly depends on the chosen filtration. In the minimal filtration such that Z is a stopping point (which does not satisfy the conditional independence property, except in some degenerate situations), one obtains [22]

$$\bar{M}_z = \int_{(0,0)}^{z} I_{\{u \le S \text{ or } v \le T\}}(dF(u,v))/(1 - F(u^-,v^-)).$$

In the product filtration (which satisfy the conditional independence property if and only if S and T are independent), one obtains [2]:

$$\bar{M}_z = \int_{(0,0)}^{z\wedge Z} (dG(\xi))/(G(\xi^-)).$$

In both cases, $M - \bar{M}$ is a weak martingale. Moreover if F is continuous, then $\bar{M}$ is continuous and M is quasi-continuous to the left.

We will finish this section with the representation theorem. This theorem is valid only for strictly simple point processes since the method of proving it is by considering the truncated point process as a multivariate one-parameter point process, where the mark is the value of the process at the second coordinate, and then to use the one-parameter multivariate point process representation theorem due to Jacod [11]. This following representation theorem was proved by Merzbach and Nualart in [27].

Theorem 2.9: Let M be a strictly simple point process satisfying the following: $M_z < \infty$ for every $z \in \mathbb{R}_+^2$, a.s.; the filtration $\{F_z\}$ is the natural filtration generated by M and verifies the conditional independence property, $M^{(1)}$ is continuous in the first coordinate, and $M^{(2)}$ is continuous in the second coordinate. Suppose that $N = \{N_z, z \in \mathbb{R}_+^2\}$ is a martingale with respect to the filtration $\{F_z\}$ which is bounded in any rectangle $R_z$. Then, there exist two processes $X_z$ and $Y(z,z')$ verifying the following properties:

(i) $X_z$ is one-predictable and adapted, and $Y(z,z')$ is $\hat{P}$-measurable. Also,

$$\int_0^s\int_0^t |X_z| M^{(1)}(dz) < \infty,$$

and

$$\int_{R_z}\int_{R_z} |Y(\xi,\xi')| M^{(1)}(d\xi) M^{(2)}(d\xi') < \infty$$

for all $z, z' \in \mathbb{R}_+^2$.

(ii) For any z such that $N_z < \infty$, we have

$$N_z = N_{0,0} + \int_{R_z} X_\xi (M^{(1)}(d\xi) - M(d\xi)) + \int_{R_z}\int_{R_z} Y(\xi,\xi')(M^{(1)}(d\xi) - M(d\xi)) \cdot (M^{(2)}(d\xi') - M(d\xi'))$$

Remarks:

1. Using the continuity properties of the processes $M^{(1)}$ and $M^{(2)}$ (on the first and second coordinates, respectively), it can be proved

that the processes $X_z$ and $Y(\zeta,\zeta')$ appearing in the above representation are essentially unique. From this fact it follows that the representation result holds for locally bounded martingales.

Actually, the boundedness property has only been used to assume $M$ has a right-continuous and left-limited version and also to check the integrability conditions (i) and (ii).

2. Clearly, a symmetric version of the representation theorem could also be stated. If we assume that $M^{(1)} = M^{(2)}$ then both representations coincide and the process $X_z$ is predictable.

## 3. Measures, orderliness and stationarity

A useful tool for the study of point processes is the following bimeasure (that is, a function of two variables such that it is a measure in each variable when the second variable is fixed). Let $M$ be a point process and denote by $\lambda_M$ (or simply by $\lambda$ when there is no risk of confusion) the bimeasure on the product space $(\Omega \times \mathbb{R}_+^2, \mathcal{F}\otimes\mathcal{B})$ defined by

$$\lambda_M(F,B) = \int_F M(B)\,d\mathbb{P},$$

This bimeasure is sometimes called the Campbell measure or the Doleans measure associated with the process $M$.

Note that generally a bimeasure cannot be extended to a measure on the $\sigma$-algebra generated by the product space (look, for example, at the Brownian sheet or even at the Brownian motion in which there is no extension). However, if $M$ is a point process, since it is increasing and non-negative, it is easy to see that $\lambda_M$ can be extended to a measure on the product space $(\Omega \times \mathbb{R}_+^2, \mathcal{F}\otimes\mathcal{B})$.

We shall suppose from now on that $\lambda$ is a Radon measure (that is finite on bounded sets, $M$ has finite expectation or is integrable). The measure $\lambda(\Omega,\cdot)$ is called the measure intensity (or sometimes the principal measure) of the point process.

In the classical theory of point processes orderliness is loosely speaking the property that points are distinct or that probabilistically they are not infinitesimally close. Various definitions have been proposed and extensively studied by D.J. Daley [7] in the real case. Notice that the word "orderliness" is used here because this condition implies that almost surely there exists an essentially unique ordering of the jump points of the process.

An interesting generalization of the strictly simple property is the following:

Definition 1: A point process $N$ is almost surely m-orderly if

$$\mathbb{P}\{N(L) = 0 \text{ or } 1 \text{ for every m-null set } L \text{ in } \mathbb{R}_+^2\} = 1,$$

where $m$ is a measure, generally the Lebesgue measure in $\mathbb{R}_+{}^2$. Note that the Poisson process in the plane is almost surely m-orderly (where $m$ is the Lebesgue measure). Conversely, an almost surely m-orderly process is strictly simple.

Other definitions are taken from Daley [7].

Definition 2: A point process $N$ is (uniformly) Khintchin orderly if to each $z \in \mathbb{R}_+^2$ and $\varepsilon > 0$ (to each $\varepsilon > 0$), there exists $\delta \equiv \delta(z,\varepsilon)$ ($\equiv \delta(\varepsilon)$) such that

$$\mathbb{P}\{N(D) \geq 2\} < \varepsilon \cdot \mathbb{P}\{N(D) \geq 1\}$$

for $D$ a rectangle such that $z \in D$ and $m(D) < \delta$,

$$(\mathbb{P}\{N(D_z) \geq 2\} < \varepsilon \cdot \mathbb{P}\{N(D_z) \geq 1\}$$

for all rectangles $D_z$ with first point $z$ such that $m(D_z) < \delta$.)

Definition 3: A point process $N$ is ordinary if for every bounded rectangle $D$,

$$\inf \sum_i \mathbb{P}\{N(D_i) \geq 2\} = 0,$$

where the inf goes over all the finite partitions $\{D_i\}$ of $D$ into mutually disjoint subrectangles.

Definition 4: A point process $N$ is (uniformly, m-) analytically orderly if to each $z \in \mathbb{R}_+^2$,

$$\lim_{m(D)\to 0} m(D)^{-1}\mathbb{P}\{N(D) \geq 2\} = 0, \text{ where } z \in D$$

(resp.

$$\lim_{m(D_z)\to 0} \sup_{z\in\mathbb{R}_+^2} M(D_z)^{-1}\mathbb{P}\{N(D_z) \geq 2\} = 0,$$

to each L m-null set

$$\lim_{n\to\infty} m(L_n)^{-1}\mathbb{P}\{N(L_n) \geq 2\} = 0$$

where $\{L_n\}_{n=1}^{\infty}$ decreases to L).

<u>Proposition 3.1</u>: Let N be an m-analytically orderly point process and let m be a Radon measure on $\mathbb{R}_+^2$. Then N is almost surely m-orderly. In particular if m is non-atomic then N is simple, and if m is continuous with respect to Lebesgue measure then N is strictly simple.

<u>Proof</u>: Let a compact set K in $\mathbb{R}_+^2$ and $\varepsilon > 0$ be given. Since m is Radon, then it is finite on compact sets and regular. Therefore to each Borel m-null set L, there is an open neighborhood $L_k$ of L such that

$$\mathbb{P}\{N(L_k) \geq 2\} < \varepsilon m(L_k).$$

A finite number of these neighborhoods, say n, cover K and each m-null set belongs to a finite number of such neighborhoods. Now we obtain in the usualy way a partitioning of K into disjoint Borel sets $A_1, \ldots, A_n$ with

$$\mathbb{P}\{N(A_i) \geq 2\} < \varepsilon m(A_i).$$

Therefore

$$\mathbb{P}\{\text{there exists a m-null set } L \text{ such that } N(L) \geq 2\}$$

$$\sum_{i=1}^{n} \mathbb{P}\{N(A_i) \geq 2\} < \varepsilon \sum_{i=1}^{n} m(A_i) = \varepsilon m(K).$$

Since this holds for any $\varepsilon > 0$, then this probability vanishes and we obtain the m-orderly property for sets in K and therefore in the whole space by the $\sigma$-compactness property. $\square$

Other results are close to those given by Daley.

Theorem 3.2: Let N be a (uniformly, m-) analytically orderly point process. Then it is ordinary and therefore N is simple.

The proof of the first part follows that of Daley (assertion 2 in [7]) since $\mathbb{R}_+^2$ is locally compact, and the second part is similar to that of Leadbetter [16].

Relations with the Khintchin orderly property involves the following possibly infinite valued measure

$$\lambda(B) = \sup\{\sum_i \mathbb{P}\{N(B_i) > 0\},\ B_i \in \mathcal{B},\ \ B_i \text{ disjoint},\ \cup_i B_i = B\}.$$

This measure is called the parametric measure of N. It is $\sigma$-finite if the point process N is finite (a.s.) on every bounded Borel set.

Theorem 3.3: Let N be a finite and uniformly Khintchin orderly point process. Then it is Khintchin orderly and therefore it is ordinary.

Here, too, the proof is essentially the same as given by Leadbetter [16].

The following result is a generalization of Korolyuk's theorem and was proved by Belyayev. A simpler proof of the two theorems was given by Leadbetter in [16], using dissecting systems.

Theorem 3.4: Let N be a point process. Then

$$\mu(B) = E[N^*(B)] = \lambda_{N^*}(\Omega,B)$$

for every Borel set B in $\mathbb{R}^2_+$. In particular, if N is simple and if its filtration satisfies the conditional independence property, then $\mu, \lambda_N$ and $\lambda_{\bar{N}}$ coincide on $\mathcal{B}$.

Another result, which can be viewed as a converse of Theorem 3.2, is the following generalized version of Dobrushin's lemma.

Theorem 3.5: Let N be a simple point process finite on bounded Borel sets. Suppose that there exists a sequence of nonnegative real numbers $\{a_n\}$ and a function $\phi(t) \to 0$ as $t \to 0$ such that for each n and for every rectangle $D_n$ with rational endpoints and same measure depending on n,

$$a_n \le \mathbb{P}\{N(D_n) > 0\}^{-1}\mathbb{P}\{N(D_n) > 1\} \le \phi(a_n).$$

Then the point process N is uniformly Khintchin orderly and uniformly analytically orderly.

Definition: A point process N is called stationary in law if for every bounded Borel set $B_1, \ldots, B_n$ in $\mathbb{R}^2_+$, the probability law of sets $(N(B_1+z), \ldots, N(B_n+z))$ does not depend on $z (z \in \mathbb{R}^2_+)$.

Note that if N is stationary in law, then the condition of Theorem 3.4 clearly holds. Note also that under stationarity, the measure intensity $\lambda(\Omega,\cdot)$ and the parametric measure $\mu(\cdot)$ are regular invariant Borel measures in $\mathbb{R}^2_+$. Thus they are constant multiples of the Lebesgue measure $m(\cdot)$ on the plane. That is $\lambda(\Omega,B) = \lambda m(B)$, and $\mu(B) = \mu m(B)$ for every Borel set B where $\lambda$ and $\mu$ are called the intensity and the parameter of the stationary point process, respectively. It is clear that $\mu \leq \lambda$ and Korolyuk's theorem states that in general they are equal. More generally, if the measures $\lambda(\Omega,\cdot)$ and $\mu(\cdot)$ are absolutely continuous with respect to the Lebesgue measure, then their Radon-Nikodym derivatives are called respectively the intensity and the parameter of the point process.

We obtain the following Khintchin's existence theorem.

Theorem 3.6: Suppose N is a simple point process which is stationary in law. Then N is both uniformly Khintchin orderly and uniformly analytically orderly, and

$$\lim_{n\to\infty} m(D_n)^{-1} \mathbb{P}\{N(D_n) > 0\} = \lambda = \mu$$

where $\{D_n\}_n$ are rectangles such that $\{m(D_n)\}_n$ is a strictly decreasing sequence to zero. Moreover if N is also almost surely an m-orderly process, then it is m-analytically orderly.

## 4. The (doubly stochastic) Poisson process

The two-parameter doubly stochastic Poisson process was defined and studied by several authors, see for example J. Grandell [9]. In the deterministic case, the Poisson process was defined and constructed by Kingman [15] and by Neveu [31] and studied by Yor [36] and Mazziotto-Szpirglas [21]. A simple construction of the Poisson process by nonstandard analysis methods was given in Manevitz-Merzbach [17]. In fact, this process has already been used many years ago in several problems in spatial statistics.

Definition: Let $M = \{M_z, z \in \mathbb{R}_+^2\}$ be a simple point process, and let $\lambda = \{\lambda_z, z \in \mathbb{R}_+^2\}$ be a nonnegative, $F^*_{(0,0)}$-measurable and integrable process. If for all $z < z'$ and $u \in \mathbb{R}$,

$$E[e^{iuM(z,z']}|F^*_z] = \exp\{(e^{iu}-1)\int_{(z,z']}\lambda_\xi d\xi\}.$$

Then M is called a $F^*$-doubly stochastic Poisson process. If $\lambda$ is deterministic then M is called a $F^*$-Poisson process, and if moreover $\lambda$ is constant (in z) then M is called $F^*$-standard Poisson.

If the filtration is omitted, we take the filtration generated by the process. In fact the Poisson process can always be chosen right-continuous with left-limits and the filtration generated by it is right-continuous and satisfies the conditional independence property. Therefore, the Poisson process (as the classical definition) is a $F^*$-Poisson process with respect to its filtration. The Poisson process has many characterizations. Before we mention them, notice that the compensator of the doubly stochastic Poisson process M is $\bar{M} = \int\lambda_\xi d\xi$ and therefore has a density $\lambda$ which is called the intensity of the process M.

If $M$ is a Poisson process, then it is quasi-continuous to the left in the following sense: For every predictable stopping line $L$, $\int I_{\{L\}} dM = 0$, which is equivalent to saying that the dual predictable projection $M^{\sim} = \tilde{M}$ is a continuous process. This means also that for all $n$, the predictable projection of the process $I_{\{L_n\}}$ vanishes. The stopping lines $\{L_n\}$ are inaccessible (where $\{L_n\}$ are the jump lines associated with $M$). Moreover, the jump points of $M$ are not stopping points. (The proof of these facts can be found in Merzbach-Nualart [27]).

<u>Theorem 4.1</u>: Let $M$ be a strictly simple point process. $M$ is a $F^*$ doubly stochastic Poisson process if and only if its compensator $\tilde{M}$ is $F^*_{(0,0)}$-measurable and $M - \tilde{M}$ is a strong martingale.

This theorem was proved in [26], without using the conditional independence property.

In the deterministic case we have other characterizations.

<u>Theorem 4.2</u>: Let $M$ be a point process. The following assertions are equivalent.

(1) $M$ is a Poisson process.

(2) $M$ has independent increments, and there exists a nonnegative deterministic function $\lambda_z$ such that for every rectangle $D$,

$$M(D) \sim P(\int_D \lambda_z dz).$$

(3) $M$ is strictly simple, its compensator $\tilde{M}$ is deterministic and $M - \tilde{M}$ is a strong martingale.

(4) $M$ is strictly simple, its compensator $\tilde{M}$ is deterministic, $M - \tilde{M}$ is a martingale and the filtration satisfies the conditional independence property.

This theorem was essentially proved by J. Mecke and by F. Papangelou [33], but a simpler proof using martingale theory can be found in [26].

Theorem 4.3: Let M be a point process. The following assertions are equivalent.

(1) M is standard Poisson.

(2) M has independent increments and there exists a positive constant $\lambda$ such that for each rectangle D, $M(D) \sim P(\lambda m(D))$.

(3) M is strictly simple, $M_z - \lambda st$ is a strong martingale for some constant $\lambda$, and every $z = (s,t)$.

(4) M is strictly simple, $M_z - \lambda st$ is a martingale for some constant $\lambda$ and every $z = (s,t)$ and the filtration satisfies the conditional independence property.

(5) There exists a positive constant $\lambda$ such that for every rectangle D with first point d, we have

$$\mathbb{P}\{M(D) = 1 \mid F_d^*\} = \lambda m(D) + o(m(D)), \quad \mathbb{P}\{M(D) > 1 \mid F_d^*\} = o(m(D)).$$

(6) The jump points $\{Z_n\}$ of M have polar coordinates $(\theta_n, r_n)$ where $\theta_n$ are i.i.d. with $\theta_n \sim U[0,2\pi]$, $r_0$, $r_n$ independent such that there exists a constant $\lambda$ satisfying $\pi(r_n^2 - r_{n-1}^2) \sim Exp(\lambda)$.

(7) M is strictly simple, has independent increments, and for every B finite union of rectangles, $\mathbb{P}\{M(B) = 0\} = e^{-\lambda m(B)}$ where $\lambda$ is a positive constant.

(8) M is strictly simple, has independent increments, M is a.s. finite on every bounded Borel set and for each $z \in \mathbb{R}_+^2$, $\mathbb{P}\{M(\{z\}) = 0\} = 1$

(9) M has independent increments and for any Borel set B, there exists Borel sets $\{B_n\}_{n=1}^{\infty}$ covering B such that for any $z \in B$, we have $M(\cap_{z \in B_n} B_n) \leq 1$ and $M(\cap_{z \in B_n} B_n) = 0$ $\mathbb{P}$ a.s.

(10) For every $z \in \mathbb{R}_+^2$, the distribution of the point process $M + I_{[z,\infty)}$ is the Palm measure $\mathbb{P}_z$ at the point z.

This is a very heterogeneous theorem, but shows several different approaches to the Poisson process. The four first assertions are proved as in Theorem 4.2 . (5) is a classical result (see for example the book of Cox-Isham). (6) is also a simple property which intervene in some geometric problems. (7) is a particular case of (8) which was proved by Prekopa. (9) is a generalization proved by Brown-Kupka [6]. (10) is an interesting application of Palm measures. It was proved by Jagers [12].

Let us now study the problem of transforming a two-parameter point process into a Poisson process by means of a two-dimensional time change. In the one-parameter case, it was done by P.A. Meyer [29] and extensively studied by F. Papangelou [32], by a family of stopping times. As it was shown in [26], a point process cannot be time changed in a Poisson process by a family of stopping points. Nevertheless, it was proved in [26] that it is possible to do it by a family of stopping lines.

<u>Theorem 4.4</u>: Let M be a strictly simple point process such that $M - \overline{M}$ is a martingale and $\overline{M}$ have a density $\lambda_z$. Suppose that the conditional independence property is satisfied. Suppose also that the function $s \to \int_0^t \lambda_{s,u} du$ is non-decreasing for all $t > 0$ and tends to infinity with t (or the same after exchanging s and t). Then there

exists a family of stopping lines $\{L_z, z \in \mathbb{R}_+^2\}$ such that $M(L_z)$ is a standard Poisson process.

The next result related to Poisson processes is the representation theorem for Poisson processes, proved by Yor [36]. Although it is a particular case of our general representation theorem (Theorem 2.9), it is interesting for itself.

<u>Theorem 4.5</u>: Let X be a square integrable martingale with respect to a filtration generated by a standard Poisson process M. Then there exists a constant $X_0$, a square integrable process $\phi = \{\phi_z, z \in \mathbb{R}_+^2\}$ and a square integrable $\hat{P}$-predictable process $\{\psi_{z_1,z_2} z_1, z_2 \in \mathbb{R}_+^2\}$ such that for every z

$$X_z = X_0 + \int_{(0,0)}^{z} \phi_\xi (dM_\xi - \lambda d\xi) + \int_{(0,0)}^{z} \int_{(0,0)}^{z} \psi_{z_1,z_2} (dM_{z_1} - \lambda dz_1)(dM_{z_2} - \lambda dz_2);$$

and this decomposition is unique.

Let us also mention in this section results in convergence of point processes to Poisson processes, which were proved by Ivanoff [10].

<u>Theorem 4.6</u>: Let $\{M^{(n)}\}_n$ be a sequence of simple point processes, such that the sequence of compensators $\bar{M}^{(n)}$ are continuous and satisfy $\bar{M}_z^{(n)} \overset{p}{\to} \bar{M}_z$ for each $z \in \mathbb{R}_+^2$ where $\bar{M}$ is a continuous deterministic function. If $\{M_z^{(n)}\}$ and $\{\bar{M}_z^{(n)}\}$ are both uniformly integrable for each $z \in \mathbb{R}_+^2$, or if there exists a bounded sequence of stopping lines $\{L_n\}$ such that $\{\bar{M}^{(n)}(L_n)_z\}$ is uniformly bounded, then $M_n \underset{n\to\infty}{\to} M$ in distribution, where M is a Poisson process with compensator $\bar{M}$.

We conclude this section by computing the laws of the stopping lines $\{L_n\}$ associated with a Poisson process. Let $\ell$ be a stepped line of separation with n exposed points: $z_1, \ldots, z_n$. In S, the Hausdorff metric is well defined; it is the same that the metric defined between the compact subsets of $\mathbb{R}_+^2$. In other words, if $\ell'$ is another line of separation, we have

$$d(\ell,\ell') = \sup_{z\in\mathbb{R}_+^2} |d(z,\ell) - d(z,\ell')| = \max\{\sup_{z\in[0,\ell]} d(z,(0,\ell']), \sup_{z\in[0,\ell']} d(z,[0,\ell])\},$$

where $d(z,A) = \inf_{z'\in A} d(z,z')$ and this last term is the "sup" distance in $\mathbb{R}_+^2$. Denote $V(\ell,h) = \{\ell' : \ell'$ line of separation: $\ell \le \ell'$ and $d(\ell,\ell') \le h\}$. It is an interval in S of length h.

Proposition 4.7: Let $\{L_n\}_n$ be the jump stopping lines of a Poisson process in $\mathbb{R}_+^2$ with intensity $\lambda$. Then

(i) $\mathbb{P}\{L_1 \in V(\ell,h)\} = e^{-\lambda([0,\ell])} \prod_{i=1}^{n} (1 - e^{-h^2})$

(ii) $\mathbb{P}\{L_n \in V(\ell,h) \mid L_{n-1} = \ell'\} = e^{-\lambda([\ell',\ell])} \prod_{i=1}^{n} (1 - e^{-h^2})$

(iii) $\mathbb{P}\{\exists k, L_k < L_{\{z_n, n\in\mathbb{N}\}}\}$

$$= 1 - \lim_{n\to\infty} \left( \sum_{k=1}^{n} e^{-|z_k|} - \sum_{i<j}^{n} e^{-|R_{z_i} \cup R_{z_j}|} + \ldots + (-1)^n e^{-|\bigcup_{k=1}^{n} R_{z_k}|} \right).$$

Moreover, these probabilities characterize the laws of $\{L_n\}_n$ where $\ell$ and $\ell'$ are stepped lines of separation with a finite number of exposed points. The proof of this proposition was given by Mazziotto and Merzbach in [19].

## 5. Palm measures

Intuitively, the Palm measure of a process is the conditional distribution of the process given that the process has a jump at the origin or more generally at a fixed point z. If N has an atom at z, the elementary definition works:

$$\mathbb{P}\{N(B) \in I / N(\{z\}) > 0\} = \mathbb{P}\{N(B) \in I, N(\{z\}) > 0\} / \mathbb{P}\{N(z) > 0\}$$

where B is a Borel subset of $\mathbb{R}_+^2$ and I a set of integer numbers. More generally, write $\lambda^*(B) = \lambda^*(\Omega, B) = E[N^*(B)]$ and assume that it is a Radon measure. Using the Radon-Nikodym theorem, for every fixed point z (even if z is not an atom of N), the above expression makes sense; and this measure is called the Palm measure of N at point $z : \mathbb{P}_z$.

The following two results which were proved by P. Jagers [12] show the importance of the notion of Palm measure.

<u>Theorem 5.1</u>: The distribution of a point process N is uniquely determined by its Palm measures $\{\mathbb{P}_z, z \in \mathbb{R}_+^2\}$ and $\lambda^*(\cdot)$.

<u>Theorem 5.2</u>: Assume that $\lambda^*$ is a Radon measure. Then N is a Poisson process if and only if for every $z \in \mathbb{R}_+^2$, the distribution of the point process $N + I_{[z,\infty)}$ is the Palm measure $\mathbb{P}_z$. (It is property 10 of Theorem 4.3).

In fact, three different definitions can be used in order to define in a precise manner the concept of Palm measure in $\mathbb{R}$. The first two were already generalized to a more general space, see for example Neveu's lectures [31] which give a good account of the different definitions and their relationships. The more natural definition, very important in several applications, defines the Palm measure by the translation of the probability of the greatest jump point which is smaller than the origin. It is the reason for which this approach was studied only in the classical case $\mathbb{R}$ in which a natural complete order is given.

Nevertheless, if we look at the jump lines, instead of at the jump points, associated with a point process defined in the whole plane, we obtain a linear order which permits the study of this natural approach.

Let $(G,+)$ be a topological group, and $\mathcal{G}$ the $\sigma$-field of its Borel sets.

<u>Definition</u>. A (measurable) flow indexed by G on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ is a family $\{\theta_t, t\in G\}$ of functions $\theta_t:\Omega\to\Omega$ such that

(i) $\theta_s\circ\theta_t=\theta_{s+t}$, and $\theta_0=$ identity,

(ii) The function $(t,\omega)\to\theta_t(\omega)$ is a measurable function from $(G\times\Omega,\mathcal{G}\otimes\mathcal{F})$ into $(\Omega,\mathcal{F})$.

(iii) $\theta_t\circ\mathbb{P}=\mathbb{P}$ for every $t\in G$.

The simplest example of flow is the translation group $\{\tau_t, t\in G\}$ defined in G by $\tau_t(s)=s-t$.

<u>Definition</u>: Let $N=\{N_t, t\in G\}$ be a point process parameterized by G. N is called stationary if there exists a flow $\{\theta_t, t\in G\}$ such that $N(\theta_t\omega,\cdot)=\tau_t[N(\omega,\cdot)]$.

Clearly a stationary point process is stationary in law.

The following theorem, proved already by Neveu [31] and others in a general framework, gives a general definition of the Palm measure.

Theorem 5.3: To any point process $N = \{N_t, t \in G\}$ which is stationary for a flow $\{\theta_t, t \in G\}$, we can associate a unique $\sigma$-finite measure $\mathbb{P}_0$ on $(\Omega, \mathcal{F})$ such that

$$E[\int_G f(\theta_t(\omega),t)dN_t] = \int\int_{\Omega G} f(\omega,t)dm\, d\mathbb{P}_0$$

for every measurable function $f:\Omega\times G \to \mathbb{R}_+$, m being here the Haar measure on the group G.

This formula is sometimes called Campbell's formula. For example, we obtain that

$$\mathbb{P}_0(\Omega) = E[N(F)]/\lambda(F)$$

for any $\lambda$-non-null Borel set F in G. More particularly if G is a discrete subset of $\mathbb{R}^n$ and N is simple, we obtain the intuitive relation:

$$\mathbb{P}_0(\Omega) = E[N(\{0\})] = \mathbb{P}\{N(\{0\}) \neq 0\}.$$

Other characterizations of Palm measure and supplementary properties can be found for example in Neveu [31].

Now, coming back to the two-parameter case, we suppose from now on that all the processes are "infinite in each quadrant", i.e. $N(\mathbb{R}^2_+) = N(\mathbb{R}^2_{+-}) = N(\mathbb{R}^2_{-+}) = N(\mathbb{R}^2_-) = +\infty$, where $\mathbb{R}^2_{+-} = \{(s,t): s > 0, t > 0\}$, $\mathbb{R}^2_{-+} = \{(s,t): s < 0, t > 0\}$ and $\mathbb{R}^2_- = \{(s,t):(s,t) \leq (0,0)\}$.

Proposition 5.4: If N is a simple jump process, then it can be written uniquely by

$$N(\omega,\cdot) = \sum_{n=-\infty}^{+\infty} \varepsilon_{L_n}(\omega)$$

where $\{L_n\}_{-\infty}^{+\infty}$ are step stopping lines satisfying

$$\ldots < L_{-1} < L_0 \le (0,0) < L_1 < L_2 < \ldots \quad \text{and} \quad \lim_{n\to\pm\infty} L_n = \pm\infty.$$

Proof: If the process was defined only in the positive quadrant of the plane, then a unique representation was constructed in [27] and also in [19] in a very general setting. Here, consider first the square $[-(n,n),(n,n)]$ and by the same method we obtain a finite strictly increasing sequence $\{L_i^{(n)}\}_{i=m(n)}^{m'(n)}$ of stopping lines which charges all the jump points of the process in the square:

$$N = \sum_{i=m(n)}^{m'(n)} \varepsilon_{L_i^{(n)}},$$

where $m(n) \le 0 \le m'(n)$. Since N is infinite in each quadrant, for n sufficiently large, some of these stopping lines are passing through the negative quadrant $\mathbb{R}^2_-$. Therefore, we can suppose (up to a new indexation) that $L_0^{(n)} \le (0,0)$ and $L_0^{(n)}$ is the greatest stopping line of the sequence with this property. Tending n to infinity, we obtain the required properties of the proposition. □

Let $S_f$ be the set of the sets of separation in $\mathbb{R}^2$ constituted by a finite number of intervals parallel to the axes. Each element of $S_f$ is of the form $\vee_{i=1}^n \bar{z}_i$, and therefore can be written uniquely as a finite sequence $\{z_i\}_{i=1}^n$ of $\mathbb{R}^2$. Now, we can consider the group of translations on $S_f$. This operation is uniquely extended to the set $S_d$: The sets of separation in $\mathbb{R}^2$ constituted by a denumerable number of intervals parallel to the axes. Indeed since $\mathbb{R}^2$ is a locally compact Hausdorff space with a countable basis, a topological group and a topological lattice, then the set $C(\mathbb{R}^2)$ of all the closed subsets of $\mathbb{R}^2$ can be provided by a topology such that $C(\mathbb{R}^2)$ is a compact Hausdorff and separable space. Moreover, it was proved in [19] that the set S of all the sets of separation is closed in $C(\mathbb{R}^2)$ and the natural order in S implies that S is in fact a topological lattice.

Using a convergence theorem of Matheron [18] we verify that the addition operation is continuous in $C(\mathbb{R}^2)$ and therefore can be uniquely extended to S. In fact, the definition of the addition between elements of $S_d$ and $\mathbb{R}^2$ will be enough for our purpose and can be defined directly.

Since $\mathbb{R}^2$ is a topological group we can consider flows $\{\theta_z, z \in \mathbb{R}^2\}$ indexed by $\mathbb{R}^2$. Furthermore, if Z is a measurable random point, then the function $\theta_Z : \Omega \to \Omega$ defined by $\theta_Z(\omega) = \theta_{Z(\omega)}(\omega)$ is measurable. The main result is the following theorem which characterizes the Palm measure by the stopping lines associated with the process. It was proved in [24].

<u>Theorem 5.5</u>: Let $N = \Sigma_{-\infty}^{+\infty} \varepsilon_{L_n}$ be a stationary strictly simple point process on $\mathbb{R}^2$ which is infinite in each quadrant, and denote by $\hat{\Omega}$ the measurable subset of $\Omega$ defined by $\hat{\Omega} = \{\omega : N(\omega, \{(0,0)\}) \neq 0\}$ $= \{\omega : L_0(\omega) \cap (\overline{(0,0)} \setminus \{(0,0)\}) \neq \phi,\ (0,0) \in L_0(\omega)\}$. Let $Z_0 \in L_0 \cap \mathbb{R}^2$ be a random jump point of N. Then the measurable function $\theta_{Z_0}$ from $\Omega$ is onto $\hat{\Omega}$ and the Palm measure $\mathbb{P}_0$ of the process N can be obtained by the relation

$$\theta_{Z_0} \circ \mathbb{P} = m(L_0, L_1)\,\mathbb{P}_0$$

which implies that $\hat{\Omega}$ is the support of $\mathbb{P}_0$.

[1] Al-Hussaini, A. and Elliott, R.J. (1981). Martingales, potentials and exponentials associated with a two-parameter jump process, Stochastics 6, 23-42.

[2] Al-Hussaini, A. and Elliott, R.J. (1981). Stochastic calculus for a two-parameter jump process, Lecture Notes in Math. 863, 233-244.

[3] Al-Hussaini, A. and Elliott, R.J. (1985). Filtrations for the two-parameter jump process, J. Multiv. Anal. 16, 118-139.

[4] Bartlett, M.S. (1974). The statistical analysis of spatial pattern, Adv. Appl. Prob. 6, 36-358.

[5] Bremaud, P. and Jacod, J. (1977). Processus ponctuels et martingales: Resultats recents sur la modelisation et le filtrage. Adv. Appl. Prob., 9, 362-416.

[6] Brown, T. and Kupka, J. (1983). Ramsey's theorem and Poisson random measures, Ann. Prob. 11, 904-908.

[7] Daley, D.J. (1974). Various concepts of orderliness for point processes. In Stochastic Geometry, ed. by Harding, E.F. and Kendall, D.E., J. Wiley & Sons, 148-161.

[8] Fisher, L. (1972). A survey of the mathematical theory of multidimensional point process. In Stochastic Point Process, Statistical Analysis, Theory and Applications, ed. Lewis, P.A., Wiley, New York, 468-513.

[9] Grandell, J. (1976). Doubly stochastic Poisson processes, Lecture Notes in Math. 529, Springer, Berlin.

[10] Ivanoff, B.G. (1985). Poisson convergence for point processes in the plane, J. Austral. Math. Soc. (Series A) 39, 253-269.

[11] Jacod, J. (1975). Multivariate point processes: predictable projection, Radon-Nikodym derivatives, Representation of Martingales, Z. Wahrsch. verw. Geb. 31, 235-253.

[12] Jagers, P. (1974). Aspects of random measures and point processes, Adv. Probab. 3, ed. Ney, M. Dekker, 179-239.

[13] Kallenberg, O. (1973). Characterization and convergence of random measures and point processes, Z. Wahrsch. verw. Geb. 27, 9-21.

[14] Kallenberg, O. (1983). Random Measures, 3rd ed., Academic Press.

[15] Kingman, J.F.C. (1967). Completely random measures, Pac. J. Math. 21, 59-78.

[16] Leadbetter, M.R. (1972). On basic results of point process theory, Proc. Sixth Berkeley Symp., 449-462.

[17] Manevitz, L. and Merzbach, E. (1986). Multi-parameter stochastic processes via non-standard analysis, submitted.

[18] Matheron, G. (1975). Random Sets and Integral Geometry, J. Wiley & Sons.

[19] Mazziotto, G. and Merzbach, E. (1986). Point processes indexed by directed sets, submitted.

[20] Mazziotto, G., Merzbach, E. and Szpriglas, J. (1981). Discontinuités des processus croissants et martingales à variation integrable, Lecture Notes in Math. 863, 59-83.

[21] Mazziotto, G. and Szpirglas, J. (1980). Equations du filtrage pour un processus de Poisson mélàngé à deux indices, Stochastics 4, 89-119.

[22] Mazziotto, G. and Szpirglas, J. (1981). Un exemple de processus à deux indices sans l'hypothèse F4. Sem. Probab. XV, Lect. Notes in Math. 850, Springer-Verlag, Berlin-Heidelberg-New York, 673-688.

[23] Mecke, J. (1967). Stationare zufallige Masse auf lokalkompakten Abelschen Gruppen. Z.F.W. 9, 36-58.

[24] Merzbach, E. (1987). Measures associated with point processes in the plane, submitted.

[25] Merzbach, E. and Nualart, D. (1985). Different kinds of two-parameter martingales, Israel J. Math. 52, 193-208.

[26] Merzbach, E. and Nualart, D. (1986). A characterization of the spatial Poisson process and changing time, Ann. Probab. 14, 1380-1390.

[27] Merzbach, E. and Nualart, D. (1987). A martingale approach to point processes in the plane, Ann. Probab., to appear.

[28] Merzbach, E. and Zakai, M. (1980). Predictable and dual predictable projections of two-parameter stochastic processes. Z. Wahrsch. verw. Geb. 53, 263-269.

[29] Meyer, P.A. (1969). Demonstration simplifee d'un theoreme de Knight, Lecture Notes in Math. 191, 191-195, Springer, Berlin.

[30] Milne, R.K. (1971). Simple proofs of some theorems on point processes, Ann. Math. Statist. 42, 368-372.

[31] Neveu, J. (1977). Processus ponctuels. Lecture Notes in Math. 598, 250-447, Springer-Verlag, Berlin-Heidelberg-New York.

[32] Papangelou, F. (1972). Integrability of expected increments of point processes and a related random change of scale, Trans. Amer. Math. Soc. 165, 483-506.

[33] Papangelou, F. (1974). The conditional intensity of general point processes and an application to line processes, Z. Wahrsch. verw. Geb. 28, 207-226.

[34] Ripley, B.D. (1976). Locally finite random sets: foundations of point process theory, Ann. Probab. 4, 983-1005.

[35] Watanabe, S. (1964). On discontinuous additive functionals and Levy measures of a Markov process, Japanese J. Math. 34, 53-70.

[36] Yor, M. (1976). Representation de martingales de carre integrable relatives aux processus de Wiener et de Poisson à n paramètres, Z. Wahrsch. verw. Geb. 35, 121-129.

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